Homework 4 Solutions

Math 131B-2

- (4.19) Let $x_0 \in [a, b]$. We claim that g is continuous at x_0. Let $\epsilon > 0$, then there is some $\delta > 0$ such that if $|x - x_0| < \delta$, $|f(x) - f(x_0)| < \epsilon$. Now suppose that $x \in (x_0, x_0 + \delta)$. Then $g(x) = \max\{f(y) : y \in [a, x]\} = \max\{\max\{f(y) : y \in [a, x_0]\}, \max\{f(y) : y \in [x_0, x]\}\}$ is either $f(x_0)$ (if $\max\{f(y) : y \in [a, x_0]\}$ is the larger of the two maxima) or is equal to $f(y)$ for some $y \in [x_0, x]$, and thus has value within ϵ of $f(x_0)$, since $|y - x_0| < \delta$. We conclude that $|g(x) - g(x_0)| < \epsilon$. Similar considerations apply when $x \in (x_0 - \delta, x_0)$, so we see that when $|x - x_0| < \delta$, we must have $|g(x) - g(x_0)| < \epsilon$. So g is continuous at x_0; since x_0 was arbitrary, g is continuous on $[a, b]$.

- (4.21) Let $f : S \to \mathbb{R}$ be continuous, and $f(p) > 0$. Let $\epsilon = \frac{f(p)}{2}$, and choose δ such that $x \in B(p; \delta)$ implies $f(x) \in B(f(p); \epsilon)$, or equivalently that $|f(p) - f(x)| < \epsilon$. In particular $f(p) - f(x) < \epsilon$, implying that $-f(x) < -f(p) + \frac{f(p)}{2}$, i.e. $f(x) > \frac{f(p)}{2} > 0$. Hence f is nonzero on $B(f(p); \epsilon)$.

- (4.25) Assume wlog that $x_1 < x_2$. Since f is continuous on $[x_1, x_2]$, it achieves a minimum somewhere on $[x_1, x_2]$. If this minimum is on (x_1, x_2) it is a local minimum and we are done. Suppose that f achieves its minimum at x_1. Then because x_1 is a local maximum, there is some neighborhood $[x_1, x_1 + \epsilon) \subset [x_1, x_2]$ such that $f(x) \geq f(x_1)$ for all $x \in [x_1, x_1 + \epsilon)$. Since $f(x_1)$ is a minimum, this implies that f is constant on $[x_1, x_1 + \epsilon)$. Ergo we have a local minimum at $x_1 + \frac{\epsilon}{2} \in (x_1, x_2)$ because the function is constant on the $\frac{\epsilon}{2}$ neighborhood $(x_1, x_1 + \epsilon)$ about $x_1 + \frac{\epsilon}{2}$. The case when f takes its minimum at x_2 is similar.

- (4.33) Consider $f(x) = \frac{1}{x}$ on $(0, \infty)$ and the Cauchy sequence $\{\frac{1}{n}\}$. Then $\{f(\frac{1}{n})\} = \{n\}$, so $f(\frac{1}{n}) \to \infty$ and in particular is not Cauchy.

- (4.30) First suppose f is continuous on S. If A is a subset of S and $x \in \overline{A}$, there is a sequence of points $\{x_n\}$ in A converging to A. Then $\{f(x_n)\}$ is a sequence of points in $f(A)$ converging to $f(x)$ by continuity, so $f(x) \in \overline{f(A)}$. Therefore $f(\overline{A}) \subset \overline{f(A)}$. Conversely, suppose we know that $f(\overline{A}) \subset \overline{f(A)}$ for all $A \subset S$. Let C be any closed set in T, and let $A = f^{-1}(C)$. Then $f(\overline{A}) \subseteq \overline{f(A)}$, but $f(A) \subseteq C$, so $\overline{f(A)} \subseteq \overline{C} = C$ since C is closed. Ergo we see that $f(\overline{A}) \subseteq C$. But this implies that $\overline{A} \subseteq f^{-1}(C) = A$. So since A contains its own closure, A is closed. Hence the preimage of any closed set in T under f is closed, implying that f is continuous.

  As an example of when this inclusion is not an equality, consider $f(x) = \tan^{-1}(x)$, which takes the closed set $A = [1, \infty)$ to $f(A) = [\frac{\pi}{4}, \frac{\pi}{2})$. Then $f(\overline{A}) = f(A) \subset [\frac{\pi}{4}, \frac{\pi}{2}] = \overline{f(A)}$.

- Question 4.
  - Suppose S is a dense subset of M. Then for $m \in M$, we consider the neighborhood $B(m; \frac{1}{n})$. Because this neighborhood is an open set, we can find $s_n \in S \cap B(m; \frac{1}{n})$. This $\{s_n\}$ is a sequence of points in S converging to m.
  - Suppose $f, g : M \to T$ are two continuous functions which agree on S. For any $m \in M$, choose a sequence $\{s_n\}$ of points in S convergin to m. Then by continuity, $\lim f(s_n) = f(m)$ and $\lim g(s_n) = g(m)$. But $\{f(s_n)\}$ and $\{g(s_n)\}$ are the same sequence, so in fact $f(m) = g(m)$ for all m.

- Question 5. Recall, from e.g. the sample midterm, that a sequence in $\mathbb{R}^n$ converges if and only if its component sequences each converge in $\mathbb{R}$. It follows immediately from our characterization of continuity in terms of sequence convergence that $f(x) = (f_1(x), \cdots, f_n(x))$ is continuous if and only if each f_i is continuous.

- Question 6. Yes, the country including its borders, viewed as a subset of the plane, is compact, and the elevation of its surface is a continuous function, so we conclude that by the extreme value theorem, there is some location in the country with elevation exactly seven thousand feet above sea level. I look forward to reading your other examples!